

Induced instability for boson-fermion mixed condensate of Alkali atoms due to attractive boson-fermion interaction

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Instabilities for boson-fermion mixed condensates of trapped Alkali atoms due to the boson-fermion attractive interaction are studied using a variational method. Three regions are shown for their instabilities according to the boson-fermion interaction strength: stable, meta-stable and unstable ones. The stability condition is obtained analytically from the asymptotic expansion of the variational total energy. The life-time of metastable states is discussed for tunneling decay, and is estimated to be very long. It suggests that, except near the unstable border, meta-stable mixed condensate should be almost-stable against clusterizations. The critical border between meta-stable and unstable phases is calculated numerically and is shown to be consistent with the Mølmer scaling condition.

Discoveries of the Bose-Einstein condensates (BEC) for Alkali atoms [1,2] and the Fermi degeneracy for trapped ^{40}K atoms [3] encourage the study for boson-fermion mixed condensates of trapped atoms. Originally, the boson-fermion condensates have been studied in bulk systems such as ^3He - ^4He and hydrogen-deuteron systems [4]. Studies of trapped mixed condensates have progressed recently; the static properties of $^{39,41}\text{K}$ - ^{40}K condensate (with repulsive boson-boson interaction) [5–7], dynamical expansions after the removal of trapping potentials [8], and the instability changes of ^7Li - ^6Li system with attractive boson-boson interaction [10].

In this paper, we study the instabilities and collapses of the polarized boson-fermion mixed condensates due to the attractive boson-fermion interaction. In BEC, such instabilities are observed in several systems: 1) the trapped meta-stable ^7Li BEC (observed experimentally [11]) equilibrated between the attractive boson-boson interaction and the kinetic pressure due to the finite confinement [12], 2) two-component uniform BEC (from bosons 1 and 2), whose stability condition is given by $g_{11}g_{22} > g_{12}^2$ (g_{ij} : the coupling constant between bosons i and j). In the latter case, the stability condition depends only on the interaction strength, but the condition for the case 1) has also the particle number dependence [12].

Let us consider the case of the trapped boson-fermion mixed condensate. In the present paper, we assume $T = 0$ and a spherical harmonic-oscillator for the trapping potential; extension to the deformed potential is straightforward. Using the Thomas-Fermi approximation for the fermion degree of freedom (appropriate for large N_f condensate), the total energy of the system becomes a functional of the boson order-parameter $\Phi(\vec{r})$ and the fermion density distribution $n_f(\vec{r})$:

$$E[\Phi(\vec{r}), n_f(\vec{r})] = \int d^3r \left[\frac{\hbar^2}{2m_b} |\nabla\Phi(\vec{r})|^2 + \frac{1}{2}m\omega^2 r^2 |\Phi(\vec{r})|^2 + \frac{g}{2} |\Phi(\vec{r})|^4 \right] \\ + \int d^3r \left[\frac{3}{5} \frac{\hbar^2}{2m} (6\pi^2)^{\frac{2}{3}} n_f^{\frac{5}{3}}(\vec{r}) + \frac{1}{2}m\omega^2 r^2 n_f(\vec{r}) + h |\Phi(\vec{r})|^2 n_f(\vec{r}) \right], \quad (1)$$

where ω and m are the angular frequency for the boson/fermion trapping potential and the boson/fermion mass ($\omega_b = \omega_f \equiv \omega$ and $m_b = m_f \equiv m$ are assumed for simplicity). In low-density gas, the boson-boson/boson-fermion coupling constants g , h are represented by the s-wave scattering lengths a_{bb} and a_{bf} : $g = 4\pi\hbar^2 a_{bb}/m_b$ and $h = 2\pi\hbar^2 a_{bf}/m_r$ where $m_r = m_b m_f / (m_b + m_f)$. In eq. (1), the fermion-fermion interaction has been neglected; the elastic fermion-fermion s-wave scattering for a polarized gas is absent because of the Pauli blocking effects and the p-wave scattering is suppressed below $100 \mu\text{K}$ [14].

Evaluating the total energy with the variational method, we take the Gaussian ansatz for the boson order-parameter:

$$\Phi(x; R) \equiv \xi^{-3/2} \phi(x; R) = \sqrt{\left(\frac{3}{2\pi}\right)^{\frac{3}{2}} \frac{N_b}{(\xi R)^3}} \exp\left(\frac{-3x^2}{4R^2}\right), \quad (2)$$

where $x = r/\xi$ is a radial distance scaled by the harmonic-oscillator length $\xi = (\hbar/m\omega)^{1/2}$, and $N_b = \int d^3x |\Phi(x; R)|^2$ is total boson number. The variational parameter R in (2) corresponds to the root-mean-square (rms) radius of the boson density distribution. This kind of variational functions have also been used in the instability investigations of

^7Li -BEC [12] and ^7Li - ^6Li system [10]. In the Thomas-Fermi approximation with (2), the fermion density distribution $n_f(\vec{r})$ becomes

$$n_f(x; R) = \frac{1}{6\pi^2} \left[\tilde{\mu}_f(R) - x^2 - \tilde{h}|\phi(x; R)|^2 \right]^{\frac{3}{2}}. \quad (3)$$

where $\tilde{h} = 2\hbar/(\hbar\omega\xi^3)$. This fermion density vanishes at the turning point Λ which is determined by

$$F(\Lambda) \equiv \tilde{\mu}_f - \Lambda^2 - \frac{C}{R^3} e^{-\frac{3\Lambda^2}{2R^2}} = 0, \quad (4)$$

where $C = (3/2\pi)^{3/2} \tilde{h} N_b$. We take $n_f(x; R) \equiv 0$ when $x \geq \Lambda$.

The scaled chemical potential $\tilde{\mu}_f = 2\mu_f/(\hbar\omega)$ in eq. (3) is determined by the normalization condition $\int d^3x n_f(x) = N_f$. As a result, the two parameters Λ and $\tilde{\mu}_f$ become functions of R .

Using eqs. (2) and (3), the total energy $E(R) = E[\Phi(x; R), n_f(x; R)]$ becomes

$$E(R)/(\hbar\omega) = N_b \left\{ \frac{9}{8} \frac{1}{R^2} + \frac{1}{2} R^2 + 2^{-5/2} \frac{3}{\pi} \tilde{g} N_b \frac{1}{R^3} \right\} + N_f \{ I_{ke}(R) + I_{ho}(R) + C R^{-3} I_{bf}(R) \}, \quad (5)$$

where $\tilde{g} = 2g/(\hbar\omega\xi^3)$ and integrals $I_{ke,ho,bf}(R)$ are defined by

$$\begin{aligned} I_{ke}(R) &= \frac{1}{5\pi N_f} \int_0^\Lambda dx x^2 F(x)^{5/2}, \\ I_{ho}(R) &= \frac{1}{3\pi N_f} \int_0^\Lambda dx x^4 F(x)^{3/2}, \\ I_{bf}(R) &= \frac{1}{3\pi N_f} \int_0^\Lambda dx x^2 e^{-\frac{3x^2}{2R^2}} F(x)^{3/2}. \end{aligned} \quad (6)$$

One of the most important candidates for mixed condensate is the $^{39,41}\text{K}$ - ^{40}K (boson-fermion) system. Their scattering lengths are not well fixed at present, and different values have been reported experimentally [15–17]. In the present calculations, to estimate the qualitative stability behaviors, we take the value for $a_{bb}(^{41}\text{K})$ in ref. [15], and it gives $\tilde{g} = 0.2$ for the boson-boson interactions with $\omega = 450$ Hz. For the attractive boson-fermion interaction \tilde{h} , several negative values are taken: $\alpha \equiv |\tilde{h}|/\tilde{g} = (0 \sim 3)$. It should be noted that the interaction strength can be shifted using the Feshbach resonance phenomena [18], whose applications for K-atom have been discussed in [16,19].

In fig. 1, numerical results for $E(R)$ in (5) are plotted for $\alpha = |\tilde{h}|/\tilde{g} = 0.1, 1.0, 2.0, 2.5, 3$ (lines a-e) with $N = 10^6$, where three kinds of patterns can be read off: stable (a), meta-stable (b,c) and unstable states (d,e). In weak boson-fermion interaction, the system is stable against the variation of R , and has an absolute minimum as an equilibrium state. For the intermediate strength, it becomes meta-stable with one local minimum (lines b,c), and, in strong attractive interactions, the minimum disappears and the system become unstable (lines d,e). It should be noted that similar meta-stable states appear in the BEC with attractive interactions (^7Li) [12], but no stable states can exist in that case.

As can be found in fig. 1, the stability of mixed condensate can be judged from the small R behavior of $E(R)$: positive divergence in small R suggests stable state. To obtain the stability condition, we consider analytically the asymptotic expansion of $E(R)$ in $R \ll 1$. For that purpose, the leading order terms of $\tilde{\mu}_f(R)$ and $\Lambda(R)$ in small R should be determined from eq. (4) and the normalization condition: $\int d^3x n_f(x; R) = N_f$. To evaluate them, we assume $\Lambda(R)/R \ll 1$ when $R \ll 1$, and expand the Gaussian function to the order of $(x/R)^2$: $\exp(-3x^2/2R^2) \sim 1 - (3x^2/2R^2)$. The consistency of these assumptions will be shown later, and can be also checked numerically. These assumptions makes the integral in the normalization condition evaluated analytically, and we obtain

$$\tilde{\mu} \sim - \left(\frac{3}{2\pi} \right)^{\frac{3}{2}} |\tilde{h}| N_b \frac{1}{R^3} + \sqrt{\frac{3}{2}} \left(\frac{3}{2\pi} \right)^{\frac{3}{4}} (48N_f)^{\frac{1}{3}} (\tilde{h}N_b)^{\frac{1}{2}} \frac{1}{R^{\frac{5}{2}}}, \quad (7)$$

$$\Lambda \sim \left(\frac{2}{3} \right)^{\frac{1}{4}} \left(\frac{2\pi}{3} \right)^{\frac{3}{8}} \frac{(48N_f)^{\frac{1}{6}}}{(\tilde{h}N_b)^{\frac{1}{4}}} R^{\frac{5}{4}}. \quad (8)$$

It should be noted that eq. (8) leads to $\Lambda/R \propto R^{1/4} \ll 1$ for $R \ll 1$, which shows the consistency of our assumption. Using eqs. (7) and (8), in the leading order in small R , the $E(R)$ becomes

$$E(R)/(\hbar\omega) \sim \left(\frac{3}{2\pi}\right)^{\frac{3}{2}} \frac{N_b}{2} \left\{ \frac{\tilde{g}N_b}{2^{\frac{2}{5}}} - |\tilde{h}|N_f \right\} \frac{1}{R^3} + \frac{1}{4} \frac{9}{6^{\frac{1}{6}}} \left(\frac{3}{2\pi}\right)^{\frac{3}{4}} (|\tilde{h}|N_b)^{\frac{1}{2}} N_f^{\frac{4}{3}} \frac{1}{R^{\frac{5}{2}}} + \mathcal{O}(R^{-2}), \quad (9)$$

so that, in small R region, $E(R)$ is dominated by R^{-3} -term, and its positivity gives the stability condition:

$$\alpha \equiv \frac{|\tilde{h}|}{\tilde{g}} < 2^{-5/2} \frac{N_b}{N_f}. \quad (10)$$

For the cases of fig. 1, it becomes $\alpha < 0.18$, which shows that only line (a) is for stable state. This condition is also plotted as the dashed line in fig. 2.

From the above stability condition, lines (b) and (c) in fig. 1 are found to be meta-stable, so that, in principle, the equilibrium states (at local minima $R = R_{eq}$) should collapse by the quantum tunneling effects into those of $R = 0$ through the potential barrier between $R = 0$ and R_{eq} . To study the collapses of meta-stable states, we estimate their collective tunneling life-time τ_{ct} applying the Gamow-theory of the nuclear α -decay to these states. For the order estimation of the tunneling life-time, as an approximation for $E(R)$, we use the simplified linear-plus-harmonic oscillator potential $V(R)$:

$$V(R) = \begin{cases} V_1(R) = FR - G, & (0 \leq R < R_t) \\ V_2(R) = \frac{1}{2}M\Omega^2(R - R_{eq})^2 + V_0. & (R_t \leq R) \end{cases} \quad (11)$$

As shown in appendix A, the tunneling life time τ_{ct} for $V(R)$ becomes $\tau_{ct}^{-1} = D \exp(-W)$, where D is a staying probability on barrier surface per unit time and $\exp(-W)$ is a transmission coefficient for the potential barrier. The explicit formulae for D and $\exp(-W)$ are

$$D = \frac{4\sqrt{2}\Omega}{\sqrt{\pi}} \sqrt{\frac{\Delta V}{\hbar\Omega} - \frac{1}{2}} \quad (12)$$

$$W = \frac{4}{3} \sqrt{\frac{2M}{\hbar^2}} \left(\Delta V - \frac{\hbar\Omega}{2} \right) (R_t - R_E) + \frac{(R_0 - R_t)^2}{a_{HO}^2}, \quad (13)$$

where R_E is a WKB turning point for the energy E and $\Delta V = V_1(R_t) - V_0$ is a barrier height (see fig. 3). $a_{HO} = \sqrt{\hbar/M\Omega}$ is harmonic oscillator length around equilibrium point. The derivation of eqs. (12,13) is given in appendix A.

To apply the above formula for the metastable condensates, we take $M = Nm$ and $\Omega = 2\omega$. These values have been determined to be the values in harmonic oscillator potential around the equilibrium point in the non-interacting case. For parameters (R_0, V_0) , $(R_t, V_1(R_t))$ and R_E , we use the values that can be obtained from the numerically calculated $E(R)$: the equilibrium, maximum, WKB turning points (for $E = V_0 + \hbar\Omega/2$).

For the present case of $N = 10^6$ and $\alpha = 2.0$ we obtain $R_t - R_E \sim 1$, $R_0 - R_t \sim 3$ in the unit of $\hbar/m\omega$, and $\Delta V \sim 40$ in the unit of $N\hbar\omega$ (see the line (c) in fig. 1). Using these values for eqs. (12,13), the staying probability and the transmission coefficient become $D \sim 10^8\omega$ and $W \sim 10^7$. As a result, we obtain $\tau_{ct}^{-1}/\omega \sim 10^8 \times \exp(-10^7)$, and the life-time τ_{ct} becomes very longer than, for example, that of clusterization by many-body collisions, $\sim (1 \sim 10)$ sec. Thus, meta-stable states should be really almost stable except the ones extremely close to the unstable border (such as line d). This long lifetime has originated from collectiveness of very many particles, and is consistent with that for meta-stable BEC [22]. To shorten the tunneling life-time and make meta-stable states collapse, the parametric excitation with $\omega = \omega(t)$ may give an interesting possibility, which will be discussed in another paper [23].

For unstable states (e. g. line e in fig. 1), using eqs. (7,8), the leading R -dependence of the fermion density at $x = 0$ and its rms radius are calculated: $n_f(0) \propto R^{-\frac{15}{4}}$ and $(x_f)_{rms} \propto R^{\frac{5}{4}}$, which are also checked numerically. They show that, in the collapses of metastable and unstable states, both boson and fermion distributions become localized and compressed, just like gravitational collapses of massive stars. Those collapses can also be found in the instability by the collective excitations evaluated with the sum-rule method [24], where the boson-fermion in-phase monopole excitation becomes the zero mode.

Finally, we comment about the critical condition for the unstable regions, which has originally been obtained by Mølmer [5] with the Thomas-Fermi approximations for both fermion and boson distribution in form of the scaling relation: $\hbar^2/g \propto N_f^{-p/(3p+6)}$ for $V \propto r^p$ -type trapping potential (the derivation of the scaling law by Mølmer, see appendix B). For the harmonic oscillator potential ($p = 2$), it becomes $\alpha^2 N_f^{1/6} = \text{const}/g$. In the present framework

of the variational method, the critical α between the meta-stable and unstable states have been evaluated for several N and plotted in fig. 2 (filled circles). From them, we can find that the variational results are also consistent with the Mølmer scaling law: the solid line added in fig. 2 in the unstable region.

In summary, for stability of boson-fermion mixed condensates with boson-boson repulsive and boson-fermion attractive interactions, the present variational calculations give the phase diagram with three regions: stable, meta-stable and unstable ones. They are clearly shown in fig. 2 in the case of $N_f = N_b \equiv N$ and $\omega_f = \omega_b \equiv \omega$. To estimate borders of stability regions generally, we studied their critical conditions analytically, from which we can find that the diagram should have qualitatively the similar pattern also in general cases of the mixed condensates. Finally, it should be noted that, in the boson-fermion collapsing processes of unstable and meta-stable states, the increasing boson/fermion densities make the two-body short-range interactions (e.g. p -wave scattering processes) and three-body (or higher) collisions more effective. These interactions, which are not essential for low-density condensates, may play important roles through the fermion-paired superfluid formations and clusterization into metal states by tunneling phenomena [22]. They will be discussed in a future publication [23].

APPENDIX A: LIFE TIME OF COLLECTIVE TUNNELING EFFECT IN METASTABLE REGION

To estimate a life time of metastable mixed condensates, we consider the collective tunneling effect with the collective variable R (the boson radius) and its effective energy $E(R)$ as a collective potential. The effective energy $E(R)$ is obtained from (9), and, for the metastable condensates, a typical shape of it is the line (c) in Fig. 1. We approximate this $E(R)$ with the linear-plus-harmonic-oscillator type potential eq. (11) (Fig. 3). M is inertia mass for the collective variable R . The meanings of other parameters (F , G , Ω , R_{eq} , R_t) can be read off in Fig. 3, and they are fixed in order that the potential $V(R)$ reproduce the $E(R)$ numerically obtained by (9). This approximation should be enough for the order estimation of the life time. For the kinetic term, we take

$$T = -\frac{\hbar^2}{2M} \frac{d^2}{dR^2}. \quad (\text{A1})$$

In $V(R)$, the metastable state $\psi_M(x)$ before tunneling is approximately given by the ground-state wave function in the harmonic oscillator potential $V_2(R)$:

$$\psi_M(R) = \frac{1}{\sqrt{\pi^{\frac{1}{2}} a_{HO}}} \exp \left[-\frac{(R - R_{eq})^2}{a_{HO}^2} \right], \quad (\text{A2})$$

where $a_{HO} = \sqrt{\hbar/M\Omega}$ is a harmonic oscillator length. This state has the zero-point energy $\hbar\Omega/2$ measured from V_0 . The continuum state $\psi_D(R)$ after the tunneling decay is obtained by the wave function in the linear potential $V_1(R)$, and its Schrödinger equation becomes

$$\left[\frac{\hbar^2}{2M} \frac{d^2}{dR^2} + E - (FR - G) \right] \psi_D(R) = 0. \quad (\text{A3})$$

To solve the above equation, we use the WKB approximation. It should be noticed that, because the state energy is lower than the potential maximum ($E < FR_t - G$), a turning point exist at $R_E = (E + G)/F$. Thus, the WKB connection formula should be used at R_E for the continuum state ψ_D :

$$\psi_D(R) = \frac{A}{2\sqrt{K(R)}} \sin \left[\frac{2}{3} \left(\frac{2MF}{\hbar^2} \right)^{\frac{1}{2}} (R_E - R)^{\frac{3}{2}} + \frac{\pi}{4} \right], \quad \text{for } 0 \leq R < R_E \quad (\text{A4})$$

$$\psi_D(R) = \frac{A}{2\sqrt{K(R)}} \exp \left[-\frac{2}{3} \left(\frac{2MF}{\hbar^2} \right)^{\frac{1}{2}} (R - R_E)^{\frac{3}{2}} \right], \quad R_E < R \quad (\text{A5})$$

where $A = \sqrt{2M/(\pi\hbar^2)}$ and $\hbar K(R) = \sqrt{2MF|R_E - R|}$.

Let us consider the tunneling decay rate Γ from $\psi_M(R)$ to $\psi_D(R)$. Using the golden rule of Fermi, it becomes

$$\Gamma = \frac{2\pi}{\hbar} \left| \int dR \psi_D^*(R) [V(R) - V_2(R)] \psi_M(R) \right|_{E=E_M}^2, \quad (\text{A6})$$

where $E_M = V_0 + \hbar\Omega/2$ is the energy of $\psi_M(R)$. Because $V(R) - V_2(R) = 0$ for $R_t < R$ and $\psi_M(0), \psi'_M(0) \ll 1$, eq. (A6) becomes

$$\Gamma = \frac{2\pi}{\hbar} \left(\frac{\hbar^2}{2M} \right)^2 \left[\frac{d\psi_D(R)}{dR} \psi_M(R) - \psi_D(R) \frac{d\psi_M(R)}{dR} \right]_{R=R_t}. \quad (\text{A7})$$

Using eqs. (A2, A4, A5), the tunneling rate Γ is shown to have the form $\Gamma = D \exp(-W)$: D can be interpreted as the staying probability Per unit time and $\exp(-W)$ is the transition coefficient. The explicit formulae for D and W are

$$D = \frac{4\sqrt{2}\Omega}{\sqrt{\pi}} \sqrt{\frac{\Delta V}{\hbar\Omega} - \frac{1}{2}} \quad (\text{A8})$$

$$W = \frac{4}{3} \sqrt{\frac{2M}{\hbar^2}} \left(\Delta V - \frac{\hbar\Omega}{2} \right) (R_t - R_E) + \frac{(R_0 - R_t)^2}{a_{HO}^2}, \quad (\text{A9})$$

where $\Delta V = V_1(R_t) - V_0$ is a barrier height. To represent the parameters F and G in V_1 by ΔV , we have used

$$V_1(R_t) = FR_t - G, \quad V_1(R_E) = FR_E - G = E = V_0 + \frac{1}{2}\hbar\Omega. \quad (\text{A10})$$

The tunneling life-time is defined as an inverse of decay rate: $\tau_{ct} = \Gamma^{-1}$.

APPENDIX B: CRITICAL CONDITION FOR UNSTABLE REGION

In this appendix, we rederive the Mølmer scaling law which gives the critical condition for the unstable region [5]. We assume the Thomas-Fermi approximations both for the boson and fermion density distributions. In that approximations, the density distributions are given as solutions of the Thomas-Fermi equations:

$$\tilde{g}n_b(x) + x^2 + \tilde{h}n_f(x) = \tilde{\mu}_b, \quad (\text{B1})$$

$$[6\pi^2n_f(x)]^{\frac{2}{3}} + x^2 + \tilde{h}n_b(x) = \tilde{\mu}_f, \quad (\text{B2})$$

where $n_b(x)$ is the boson density distribution scaled by the harmonic-oscillator length $\xi = (\hbar/m\omega)^{1/2}$ ($n_b(x) = |\Phi(x)|^2\xi^3$). The scaled chemical potentials, $\tilde{\mu}_{b,f}$, and coupling constants, \tilde{g} and \tilde{h} , have been defined in the main body of this paper.

Eliminating $n_b(x)$ in (B2) by (B1), we obtain the equation $F[n_f(x)] = G[n_f(x)]$ to determine the fermion density $n_f(x)$, where

$$F[n_f(x)] \equiv [6\pi^2n_f(x)]^{\frac{2}{3}}, \quad G[n_f(x)] \equiv \tilde{\mu}_f + \frac{|\tilde{h}|}{\tilde{g}}\tilde{\mu}_b - \left(1 + \frac{|\tilde{h}|}{\tilde{g}}\right)x^2 + \frac{|\tilde{h}|^2}{\tilde{g}}n_f(x). \quad (\text{B3})$$

We concentrate on the central density $n_f(0)$. In order to determine $n_f(0)$ two conditions should be satisfied:

$$F[n_f(0)] = G[n_f(0)], \quad \frac{\delta F}{\delta n_f}[n_f(0)] \geq \frac{\delta G}{\delta n_f}[n_f(0)], \quad (\text{B4})$$

Evaluating equations in (B4) with (B3), we obtain the critical condition for the unstable region:

$$\frac{|\tilde{h}|^2}{\tilde{g}} = \frac{4\pi^2}{\sqrt{3}} \left(\tilde{\mu}_f + \frac{|\tilde{h}|}{\tilde{g}}\tilde{\mu}_b \right). \quad (\text{B5})$$

To evaluate the right-hand side of (B5), we should use the relations between $\tilde{\mu}_{b,f}$ and the boson/fermion particle number $N_{b,f}$, which are obtained by solving the Thomas-Fermi equations in (B2). Here we assume that $\tilde{\mu}_b = 0$ and $\tilde{\mu}_f = 2(6N_f)^{1/3}$ ($\tilde{\mu}_f$ for a free fermion system) for (B5). Consequently, we obtain the Mølmer scaling relation:

$$\frac{|\tilde{h}|^2}{\tilde{g}} = \frac{4\pi^2}{6^{1/6}\sqrt{6}} N_f^{-1/2} \sim 12.0 N_f^{-1/6}. \quad (\text{B6})$$

It should be noted that the coefficient 12.0 in (B6) are close to the value 13.8 which is obtained by Mølmer [5] with numerically evaluating the Thomas-Fermi equations.

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FIG. 1. The total energy variation $E(R)/(N\hbar\omega)$ of the boson-fermion mixed condensate with the Gaussian boson distribution $\Phi(x; R)$ of radius R : $E(R)$ and $\Phi(x; R)$ are defined in (5, 2). The Thomas-Fermi density function (3) is applied for the fermion distribution. $N = N_b = N_f = 10^6$, $\tilde{g} = 0.2$, and $\alpha = \tilde{h}/\tilde{g} = 0.1, 1.0, 2, 2.5, 3$ for lines a-e.

FIG. 2. Stability phase diagram of the boson-fermion mixed condensate for $N \equiv N_b = N_f$ and $\alpha = \tilde{h}/\tilde{g}$ when $\tilde{g} = 0.2$. Three regions exist in it: stable (S), meta-stable (MS) and unstable (US) ones. The dashed line at $\alpha = 0.18$ corresponds to the border of stable region, and the solid line shows the Mølmer scaling law between meta-stable and unstable regions. The open circles correspond to the parameters of the states in fig. 1, and the filled circles show numerically confirmed critical states between meta-stable and unstable regions.

FIG. 3. Simplified potential for meta-stable condensates. The $V_1(R)$ and $V_2(R)$ are the linear and harmonic oscillator parts of the potential $V(R)$. The R_{eq} , R_t , R_E are the equilibrium, maximum and the WKB turning points. V_0 is the equilibrium energy of the potential.

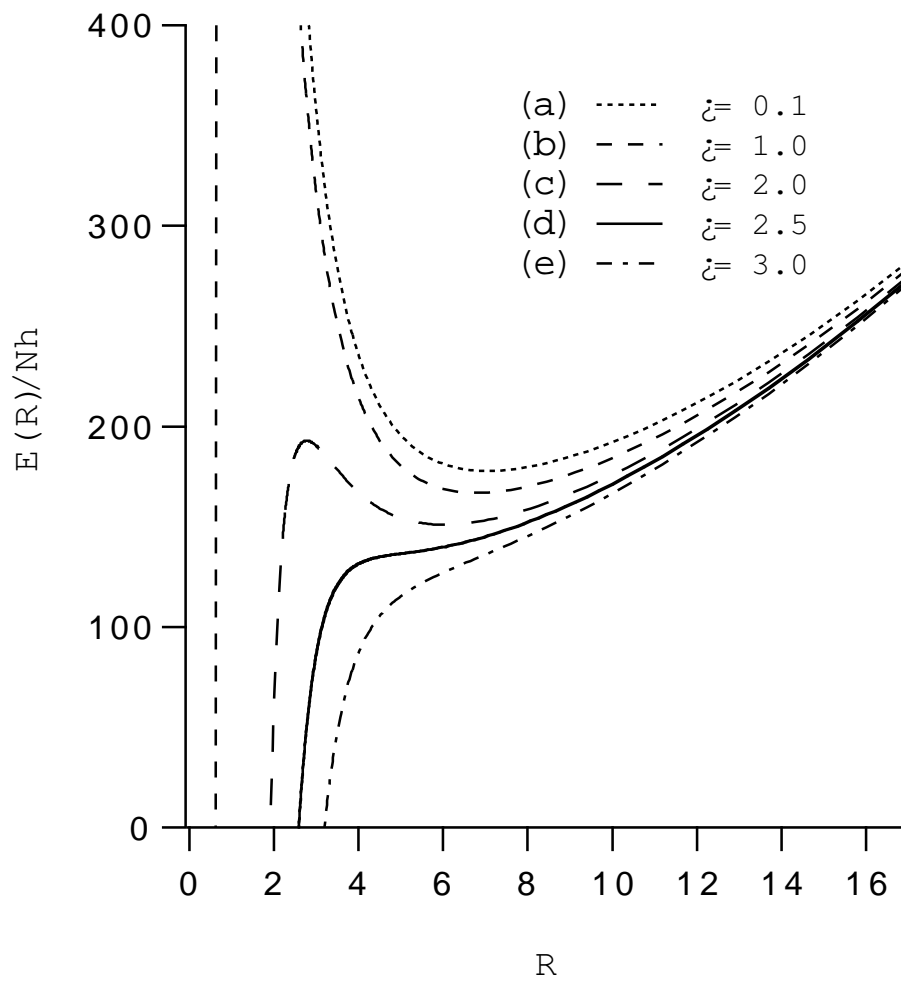


Figure 1

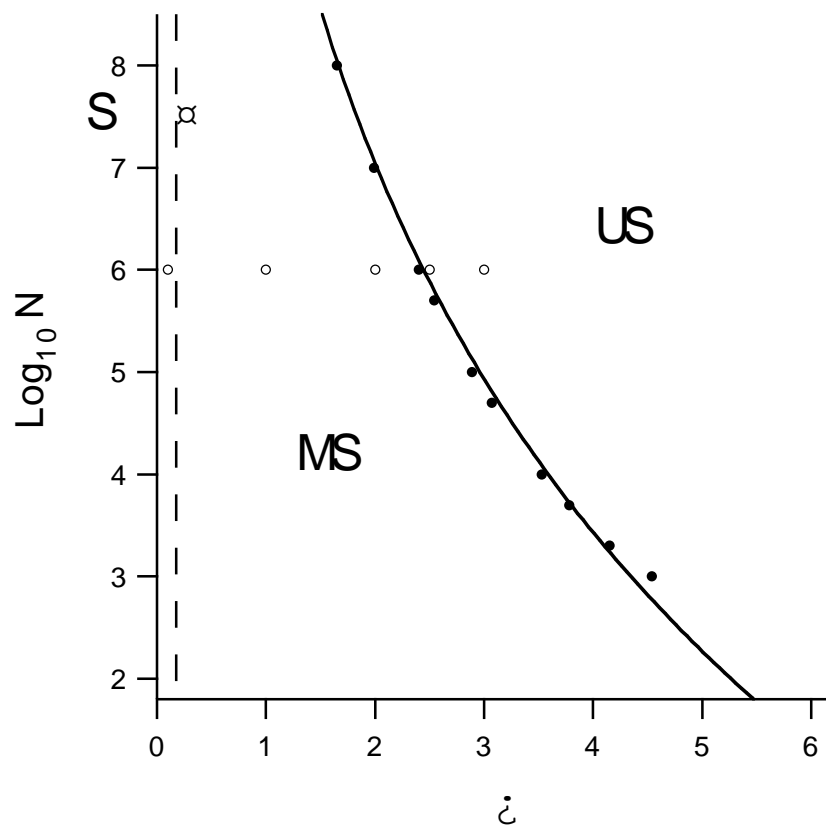


Figure 2

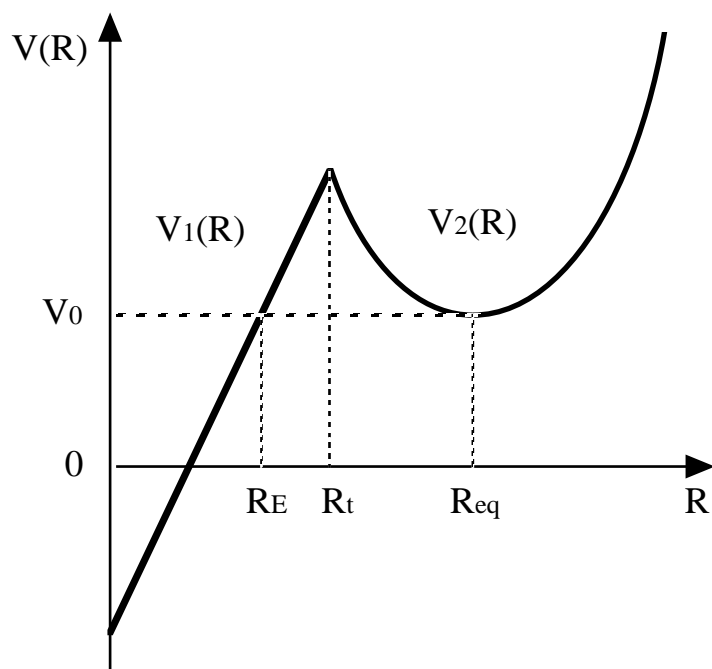


Figure 3